Published by Institute of Physics Publishing for SISSA/ISAS
Received: *April 29, 2003*
Accepted: *May 16, 2003*

Effective superpotentials via Konishi anomaly

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Abstract: We use Ward identities derived from the generalized Konishi anomaly in order to compute effective superpotentials for SU(N), SO(N) and Sp(N) supersymmetric gauge theories coupled to matter in various representations. In particular we focus on cubic and quartic tree level superpotentials. With this technique higher order corrections to the perturbative part of the effective superpotential can be easily evaluated.

Keywords: Anomalies in Field and String Theories, Supersymmetric Effective Theories.

JHEP05(2003)041

Contents

1. Introduction

The nonperturbative dynamics of supersymmetric gauge theories is a rich and fascinating subject. These theories exhibit various phenomena, such as gaugino condensation and confinement, a complete understanding of whose is still lacking. Recently a promising step toward this direction has been made by Dijkgraaf and Vafa [1, 2, 3] who conjectured that the effective superpotentials for a wide class of $\mathcal{N} = 1$ theories can be computed by summing planar diagrams in a related matrix model. Arguments supporting this conjecture were given using chiral superspace techniques in [4] and using anomalies in [5]. In particular this last result led the authors of [5] to further interesting developments [6]–[9]. Many aspects of this conjecture have been widely studied in the last months [10].

In [5], a generalization of the Konishi anomaly [11, 12] has been considered, leading to a set of identities with the same structure of the loop equations of a matrix model. Using these identities the authors were able to derive the effective superpotentials of [1] up to an integration constant (basically the Veneziano-Yankielowicz superpotential [13]). The aim of this work is to follow their approach to compute effective superpotentials in some specific cases for SU(N), SO(N) and Sp(N)[1] with matter in various representations.

[1]Here we use conventions such that N is an even number, i.e. the rank of the group is $\frac{N}{2}$.

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Some superpotentials for SO(N) and Sp(N) have already been computed in the framework of [1]–[21]. For adjoint matter, the results obtained reflect the charge of the orientifold plane used in the geometric engineering of the gauge theory. For discussions on SU(N) see for example [22].

This paper is organized as follows: in the second section we review some results from [5] and in section 3 we compute the generalized Konishi anomaly for other groups, namely SO(N), Sp(N) and SU(N) and write down the equations (analog to the loop equations of the matrix model) we will use to compute effective superpotentials. In the fourth section we perform explicit computations of effective superpotentials for cubic and quartic interactions with matter in various representations. Finally in the last section we discuss our results and propose some further developments.

2. A Fairy Tale of an Anomaly

In this section we will briefly summarize some results of [5].

Let us begin with an U(N) theory with matter Φ in the adjoint representation and a tree level superpotential

$$W(\Phi) = \sum_{k=0}^{n} \frac{g_k}{k+1} \operatorname{Tr} \Phi^{k+1} \tag{2.1}$$

as in [5]. This theory has a natural ring structure, the chiral ring, defined by the equivalence classes of gauge invariant chiral operators modulo $\{\overline{Q}_{\dot\alpha}, \ldots\}$. It can be proven that it is generated by operators of the form $\operatorname{Tr}\Phi^k$, $\operatorname{Tr}\Phi^k W_\alpha$, and $\operatorname{Tr}\Phi^k W_\alpha W^\alpha$, where W^α is the gauge superfield. An important element of the chiral ring is the glueball superfield $S = -\frac{1}{32\pi^2} \operatorname{Tr} W_\alpha W^\alpha$, which is believed to describe the low energy dynamics of the theory. Given a tree level superpotential, the effective superpotential as a function of the glueball superfield S can be found, restricting ourselves to the chiral ring, by means of Ward identities following from a generalization of the Konishi anomaly.

In [5] the most general variation of Φ in the chiral ring was considered

$$\Phi = f(\Phi, W_\alpha) \tag{2.2}$$

where f is a general holomorphic function, leading to the generalized Konishi anomaly[2]

$$\overline{D}^2 J_f = \operatorname{Tr} f(\Phi, W_\alpha) \frac{\partial W(\Phi)}{\partial \Phi} + \sum_{ijkl} A_{ij,kl} \frac{\partial f(\Phi, W_\alpha)_{ji}}{\partial \Phi_{kl}} \tag{2.3}$$

where[3]

$$J_f = \operatorname{Tr} \overline{\Phi} e^{\mathrm{ad}\ V} f(\Phi, W_\alpha)$$

and

$$A_{ij,kl} = \frac{1}{32\pi^2} \left[W_\alpha, [W^\alpha, T_{lk}]\right]_{ij} \tag{2.4}$$

[2]Here and in the following we will consider $W(\Phi)$ as a matrix every time it appears inside a trace.

[3]With ad V we mean the adjoint representation: $(\mathrm{ad}\ V\ \Phi)^i{}_j = V^i{}_k \Phi^k{}_j - \Phi^i{}_k V^k{}_j$.

T_{lk} being the generators of the gauge group (U(N) in this case). Note that at this point this is quite general and a change in the gauge group will reflect only in the explicit form of the generators T_{lk}. For U(N) we have $(T_{lk})_{ij} = (e_{lk})_{ij} = \delta_{il}\delta_{jk}$ and

$$\overline{D}^2 J_f = \mathrm{Tr}\, f(\Phi, W_\alpha) \frac{\partial W(\Phi)}{\partial \Phi} + \frac{1}{32\pi^2} \sum_{i,j} \left[W_\alpha, \left[W^\alpha, \frac{\partial f}{\partial \Phi_{ij}} \right] \right]_{ji} . \tag{2.5}$$

Finally, taking the vacuum expectation value, we find

$$\left\langle \mathrm{Tr}\, f(\Phi, W_\alpha) \frac{\partial W}{\partial \Phi} \right\rangle = -\frac{1}{32\pi^2} \left\langle \sum_{i,j} \left(\left[W_\alpha, \left[W^\alpha, \frac{\partial f(\Phi, W_\alpha)}{\partial \Phi_{ij}} \right] \right] \right)_{ji} \right\rangle . \tag{2.6}$$

Now, we define

$$\begin{aligned} T(z) &= \sum_{k \geq 0} z^{-1-k}\, \mathrm{Tr}\, \Phi^k = \mathrm{Tr}\, \frac{1}{z - \Phi} \\ R(z) &= -\frac{1}{32\pi^2}\, \mathrm{Tr}\, W_\alpha W^\alpha \frac{1}{z - \Phi} . \end{aligned} \tag{2.7}$$

Taking

$$\delta \Phi_{ij} = f_{ij}(\Phi, W_\alpha) = R(z)_{ij} \tag{2.8}$$

using the Konishi anomaly and the algebraic relation

$$\sum_{i,j} \left[\chi_1, \left[\chi_2, \frac{\partial}{\partial \Phi_{ij}} \frac{\chi_1 \chi_2}{z - \Phi} \right] \right]_{ij} = \left(\mathrm{Tr}\, \frac{\chi_1 \chi_2}{z - \Phi} \right)^2 \tag{2.9}$$

which holds if $\chi_1^2 = \chi_2^2 = 0$ and $[\Phi, \chi_\alpha] = 0$, one can obtain

$$R^2(z) = W'(z) R(z) + \frac{1}{4} f(z) \tag{2.10}$$

$f(z)$ being a polynomial of degree $n - 1$ in z. Solving this equation, we obtain

$$2R(z) = W'(z) - \sqrt{W'(z)^2 + f(z)} . \tag{2.11}$$

Analogously, by taking

$$\delta \Phi_{ij} = f_{ij}(\Phi) = T(z)_{ij} \tag{2.12}$$

one finds

$$2R(z) T(z) = W'(z) T(z) + \frac{1}{4} c(z) \tag{2.13}$$

where $c(z)$, like $f(z)$, is a polynomial of degree $n - 1$. This equation can be used together with (2.11) in order to derive a closed equation for $T(z)$

$$T(z) = -\frac{1}{4} \frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} . \tag{2.14}$$

As explained in [5], expanding (2.14) in powers of $\frac{1}{z}$ we can extract the vacuum expectation values of the operators $\mathrm{Tr}\,\Phi^k$ that can be integrated to obtain the effective superpotential up to a constant of integration independent of the couplings (but in general dependent on S), using the relation

$$\frac{\partial W_{\rm eff}}{\partial g_k} = \left\langle \mathrm{Tr}\,\frac{\Phi^{k+1}}{k+1} \right\rangle . \tag{2.15}$$

Then, the general strategy is to write (2.5) for the gauge group under consideration, obtain the generalization of (2.10) and (2.13), solve for $T(z)$ as in (2.14) and finally extract the superpotential (as explained before).

The information about the chosen vacuum is encoded in the explicit form of the denominator of equation (2.14). In general, the choice of the function $f(z)$ in the curve

$$y^2 = W'(z)^2 + f(z)$$

determines how the gauge group is broken and selects the vacuum. In particular, for the case of unbroken gauge group, only one of the n zeros of $W'(z)$ splits in a branch cut (around the vacuum), while the others will only get shifted. This means that the curve factorizes as

$$y^2 = W'(z)^2 + f(z) = Q(z)^2\,(z+\alpha+\beta)\,(z+\alpha-\beta)$$

where the $n-1$ unsplitted zeros of y are contained in the polynomial $Q(z)$.

3. The Konishi anomaly for other groups

In this section, we will derive the Konishi anomaly and the equation for $T(z)$ for SO(N) (in some detail) and Sp(N) with matter in the adjoint and symmetric (antisymmetric for Sp(N)), both traceful and traceless, representations and finally for SU(N) with matter in the adjoint representation.

Let us begin with the case of an SO(N) gauge theory with adjoint matter and evaluate explicitly (2.3). We take the generators of SO(N) to be $T_{lk} = (e_{lk} - e_{kl})$ with $(e_{lk})_{ij} = \delta_{il}\delta_{jk}$. First of all, we note that the identity (2.9) holds due to the spinorial properties of χ_α and is independent of the generators up to numerical factors. As can be easily checked the equation for $R(z)$ (2.10) then becomes

$$\frac{1}{2}R^2(z) = W'(z)R(z) + \frac{1}{4}f(z)$$

whose solution is

$$2R(z) = 2W'(z) - 2\sqrt{W'(z)^2 + \frac{f(z)}{2}}\,. \tag{3.1}$$

Now let us focus on the equation for $T(z)$ (2.13) and restrict ourselves to variations of the form[4]

$$\delta\Phi = f(\Phi) = -\frac{1}{32\pi^2}\frac{1}{z-\Phi}\,. \tag{3.2}$$

[4]Properly speaking one should add also the term $\frac{1}{z+\Phi}$ since $\delta\Phi$ has to be an element of SO(N) in the adjoint representation, that is to say an antisymmetric matrix. However it can be checked that it will contribute exactly as the previous, giving only an overall factor of 2. Because of this it will be omitted in the following analysis.

Then the equation for the anomaly gives

$$\overline{D}^2 J_f = -\frac{1}{32\pi^2}\,\mathrm{Tr}\,\frac{1}{z-\Phi}\frac{\partial W(\Phi)}{\partial\Phi} + \\ +\frac{1}{32\pi^2}\frac{1}{4}\sum_{ijkl}[W_\alpha,[W^\alpha,(e_{lk}-e_{kl})]]_{ij}\left(\frac{1}{z-\Phi}(e_{kl}-e_{lk})\frac{1}{z-\Phi}\right)_{ji} \tag{3.3}$$

Let us focus on the second term on the right hand side

$$\frac{1}{32\pi^2}\frac{1}{4}\sum_{ijkl}[W_\alpha,[W^\alpha,(e_{lk}-e_{kl})]]_{ij}\left(\frac{1}{z-\Phi}(e_{kl}-e_{lk})\frac{1}{z-\Phi}\right)_{ji} = \\ = \frac{1}{32\pi^2}\frac{1}{4}\left(4\,\mathrm{Tr}\,\frac{W_\alpha W^\alpha}{z-\Phi}\,\mathrm{Tr}\,\frac{1}{z-\Phi} - 8\,\mathrm{Tr}\left(W_\alpha W^\alpha\frac{1}{z-\Phi}\left(\frac{1}{z-\Phi}\right)^T\right) - \right. \\ \left. - 4\,\mathrm{Tr}\,\frac{W_\alpha}{z-\Phi}\,\mathrm{Tr}\,\frac{W^\alpha}{z-\Phi}\right) \tag{3.4}$$

where we have used the commutation properties of the operators in the chiral ring. Now, being Φ an antisymmetric matrix, we have

$$\left(\frac{1}{z-\Phi}\right)^T = \frac{1}{z+\Phi}\,. \tag{3.5}$$

Next we use the identity

$$\frac{1}{z-\Phi}\frac{1}{z+\Phi} = \frac{1}{2z}\left(\frac{1}{z-\Phi}+\frac{1}{z+\Phi}\right)$$

in order to write

$$\mathrm{Tr}\,W_\alpha W^\alpha\frac{1}{z-\Phi}\left(\frac{1}{z-\Phi}\right)^T = \frac{1}{z}\,\mathrm{Tr}\,W_\alpha W^\alpha\frac{1}{z-\Phi}\,.$$

Taking expectation values of (3.3) and using the definitions (2.7) we have

$$W'(z)T(z) + \frac{1}{4}c(z) = R(z)T(z) - 2\frac{R(z)}{z}\,.$$

Using the relation (3.1) we finally obtain the equation for $T(z)$

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}} - \frac{2}{z}\frac{W'(z)-\sqrt{W'(z)^2+f(z)}}{\sqrt{W'(z)^2+f(z)}}\,. \tag{3.6}$$

Here we absorbed a factor of $\frac{1}{2}$ in a redefinition of $f(z)$ (we will always use this convention when speaking about $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$). As previously explained, from (3.6) we can obtain the effective superpotential for an $\mathrm{SO}(N)$ gauge theory with adjoint matter and tree level superpotential (2.1).

Now let us consider the same gauge theory but with matter in the symmetric representation (that is, Φ is now a symmetric matrix and we use a symmetric representation for the $\mathrm{SO}(N)$ basis). In this case (3.5) becomes

$$\left(\frac{1}{z-\Phi}\right)^T = \frac{1}{z-\Phi}$$

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and

$$\operatorname{Tr} W_\alpha W^\alpha \frac{1}{z-\Phi}\left(\frac{1}{z-\Phi}\right)^T = \operatorname{Tr} W_\alpha W^\alpha \left(\frac{1}{z-\Phi}\right)^2 = -\frac{d}{dz}\left(\operatorname{Tr} W_\alpha W^\alpha \frac{1}{z-\Phi}\right) \tag{3.7}$$

Again, from (2.5), (2.7) and using now (3.7) one finds

$$W'(z)T(z) + \frac{1}{4}c(z) = R(z)T(z) - 2R'(z) \tag{3.8}$$

and the equation for $T(z)$ becomes

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} - 2\frac{\frac{d}{dz}\left(W'(z) - \sqrt{W'(z)^2 + f(z)}\right)}{\sqrt{W'(z)^2 + f(z)}}. \tag{3.9}$$

To complete our discussion about SO(N), let us consider now Φ in the traceless symmetric representation. All we have to do is to take the previous results and subtract the trace of Φ. For instance, (3.2) will now become

$$\delta\Phi = f(\Phi) = -\frac{1}{32\pi^2}\left(\frac{1}{z-\Phi} - \frac{1}{N}\operatorname{Tr}\frac{1}{z-\Phi}\right). \tag{3.10}$$

This will not produce any change in (3.4) (since the trace part is proportional to the identity matrix and it is entering in the commutator); the only modifications will arise in the left hand side of (2.6) which now becomes

$$-\frac{1}{32\pi^2}\operatorname{Tr}\left(\frac{1}{z-\Phi} - \frac{1}{N}\operatorname{Tr}\left(\frac{1}{z-\Phi}\right)\right)\frac{\partial W(\Phi)}{\partial \Phi}$$

and in the equation for $R(z)$ (2.10) which now reads[5]

$$R^2(z) = \left(W'(z) - \frac{1}{N}W'(\Phi)\right)R(z) + \frac{1}{4}f(z). \tag{3.11}$$

Now equation (3.8) becomes

$$T(z)\left(W'(z) - \frac{1}{N}W'(\Phi)\right) + \frac{1}{4}c(z) = R(z)T(z) - 2R'(z).$$

Finally we can write the equation for $T(z)$ for matter in the symmetric traceless representation

$$\begin{aligned} T(z) = &-\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}} - \\ &-2\frac{\frac{d}{dz}\left(\left(W'(z) - \frac{1}{N}W'(\Phi)\right) - \sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}\right)}{\sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}}. \end{aligned} \tag{3.12}$$

[5]Remember that we are taking vacuum expectation values; properly speaking $W'(\Phi)$ has to be understood as $\langle W'(\Phi)\rangle$.

Now we will focus on an Sp(N) gauge theory with matter in the adjoint (symmetric) and in the antisymmetric (both traceful and traceless) representations. With symmetric (antisymmetric) we mean that Φ has to be considered as a matrix MJ where M is a symmetric (antisymmetric) matrix and J is the invariant antisymmetric tensor of Sp(N). We take the generators of Sp(N) as $(e_{lk} + e_{kl})$ with $(e_{lk})_{ij} = \delta_{il}\delta_{jk}$. The analysis for the Sp(N) case is almost identical to the one for the SO(N) case, the only change being the sign in the generators (and of course the different properties of the matrices representing the field Φ, since the antisymmetric invariant J will enter in the intermediate steps). Because of this we will only state our results. For matter in the symmetric representation the equation for $T(z)$ becomes

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} + \frac{2}{z}\frac{W'(z) - \sqrt{W'(z)^2 + f(z)}}{\sqrt{W'(z)^2 + f(z)}} \tag{3.13}$$

for matter in the antisymmetric traceful representation

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} + 2\frac{\frac{d}{dz}\left(W'(z) - \sqrt{W'(z)^2 + f(z)}\right)}{\sqrt{W'(z)^2 + f(z)}} \tag{3.14}$$

and finally for matter in the antisymmetric traceless representation

$$\begin{aligned} T(z) = &-\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}} + \\ &+2\frac{\frac{d}{dz}\left(\left(W'(z) - \frac{1}{N}W'(\Phi)\right) - \sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}\right)}{\sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}} \end{aligned} \tag{3.15}$$

As a last example, let us consider the SU(N) gauge group with matter in the adjoint representation. This is basically equivalent to consider an U(N) gauge theory subtracting the trace as in (3.10) (remember that the term containing the trace will not produce any modification when entering in a commutator). Then, one can easily find

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\,.\right)^2 + f(z)}} \tag{3.16}$$

4. The effective superpotential

In this section we apply the previous results in order to find the effective superpotential for SO(N), Sp(N) and SU(N) gauge theories with quartic and cubic superpotential and matter in various representations.

As already mentioned, the general strategy is to write down the equation for $T(z)$ for every particular case, expand it in powers of $\frac{1}{z}$ and extract the vacuum expectation values of the operators $\langle \mathrm{Tr}\,\Phi^k \rangle$, from where the effective superpotential can be obtained by using equation (2.15).

4.1 Quartic superpotential

$\mathrm{Sp}(N)/\,\mathrm{SO}(N)$ **with matter in the antisymmetric/symmetric representation.** Let us suppose the following tree level superpotential:

$$W(\Phi) = \frac{m}{2}\,\mathrm{Tr}\,\Phi^2 + \frac{g}{4}\,\mathrm{Tr}\,\Phi^4\,. \tag{4.1}$$

As seen in the previous section we obtain the following equation for $T(z)$

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}} + 2\epsilon\frac{W''(z) - (\sqrt{W'(z)^2+f(z)})'}{\sqrt{W'(z)^2+f(z)}}\,. \tag{4.2}$$

Where $\epsilon = \pm 1$ for $\mathrm{Sp}(N)/\,\mathrm{SO}(N)$ and $c(z)$ and $f(z)$ are polynomials of degree 2

$$f(z) = \sum_{i=0}^{2} f_i z^i$$

and

$$c(z) = \sum_{i=0}^{2} c_i z^i$$

The denominator of both terms in equation (4.2) can be factorized as explained previously. We impose (see for example [20]):

$$W'(z)^2 + f(z) = g^2\left(z^2-k^2\right)^2\left(z^2-4\mu^2\right). \tag{4.3}$$

From this condition we arrive to the following expressions:

$$\begin{aligned} k &= \sqrt{-\frac{m+2g\mu^2}{g}} \\ \mu &= \sqrt{-\frac{gm+\sqrt{g^2\left(-3f_2+m^2\right)}}{6g^2}}\,. \end{aligned} \tag{4.4}$$

Imposing condition (4.3) gives a system of equations for k and μ with a set of solutions. Note that we choose the particular k and μ tending to $\sqrt{-\frac{m}{g}}$ and 0 for f_2 going to zero (that is the classical limit). This means we place the branch cut around zero. In order to have the correct asymptotic behavior of $R(z)$ for large z ($R(z) \sim \frac{S}{z}$, see eq. (2.7)) it can be shown that[6] $f_2 = -2gS$. Similarly the correct asymptotic behavior of $T(z)$ for large z ($T(z) \sim \frac{N}{z}$) sets $c_2 = -4gN$. c_0 and c_1 can be found by asking the condition that $T(z)$ has no poles in k and $-k$ (i.e. we choose our vacuum around $\Phi = 0$ and the gauge group remains unbroken) or, equivalently:

$$\begin{aligned} \frac{1}{2\pi i}\oint_{C_k} dz\,T(z) &= 0 \\ \frac{1}{2\pi i}\oint_{C_{-k}} dz\,T(z) &= 0\,. \end{aligned} \tag{4.5}$$

[6]The difference with [5] is due to our redefinition of $f(z)$.

For the present case we obtain:

$$\begin{aligned} c_0 &= 4\left(2\epsilon\left(m+gk\left(3k-2\sqrt{k^2-4\mu^2}\right)\right)+gk^2N\right) \\ c_1 &= 0\,. \end{aligned} \tag{4.6}$$

Next, we expand $T(z)$ in powers of $\frac{1}{z}$ and obtain:

$$\frac{\partial W_{\rm eff}}{\partial m} = \frac{2\epsilon}{g}\left(m+3g\mu^2+\sqrt{(m+6g\mu^2)\,(m+2g\mu^2)}\right)+\mu^2 N\,. \tag{4.7}$$

This expression can be expanded in powers of S and integrated in order to obtain the effective superpotential up to any given order; for instance up to fourth order, it reads

$$\begin{aligned} W_{\rm eff} &= \frac{1}{2}\,(-2\,\epsilon+N)\,S\log m+\frac{g}{8m^2}\,(-10\,\epsilon+3N)\,S^2-\frac{g^2}{16m^4}\,(-38\,\epsilon+9N)\,S^3 \\ &\quad+\frac{g^3}{96m^6}\,(-662\,\epsilon+135N)\,S^4+\cdots\,. \end{aligned} \tag{4.8}$$

Several comments are in order. Having obtained $W_{\rm eff}$ integrating with respect to m, the result is correct up to a function of g and S (a part of which is the Veneziano-Yankielowicz superpotential); we could have chosen the coefficient of the term $1/z^4$ and integrated with respect to g. As the perturbative part of the potential depends only on the ratio g/m^2 by dimensional reasons [5], a function of only one of the coupling constants cannot contribute. For the same reason, from now on, we will only consider the $1/z^3$ term.

$\mathrm{Sp}(N)/\,\mathrm{SO}(N)$ **with matter in the adjoint representation.** This case is completely analogous to the case studied before. Now the equation for $T(z)$ reads:

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}}+\epsilon\frac{2}{z}\frac{W'(z)-\sqrt{W'(z)^2+f(z)}}{\sqrt{W'(z)^2+f(z)}}\,.$$

Where again $c(z)$ and $f(z)$ are polynomials of degree 2. The denominator of both terms can be factorized as before (since it is the same) and we obtain the same values for the parameters k and μ. Again $f_2=-2gS$ and $c_2=-4gN$ and the conditions (4.5) must be imposed. The values obtained for c_0 and c_1 are:

$$\begin{aligned} c_0 &= 4\left(2\epsilon\left(gk^2+m\right)+gk^2N\right) \\ c_1 &= 0\,. \end{aligned} \tag{4.9}$$

Again, expanding $T(z)$ in powers of $\frac{1}{z}$ and and extracting the coefficient of $\frac{1}{z^3}$ we obtain:

$$\frac{\partial W_{\rm eff}}{\partial m} = \mu^2(N+2\epsilon) \tag{4.10}$$

that can be expanded in powers of S and integrated with respect to m, to give:

$$W_{\rm eff} = (N+2\epsilon)\left(\frac{S}{2}\log m+\frac{3gS^2}{8m^2}-\frac{9g^2S^3}{16m^4}+\frac{45g^3S^4}{32m^6}-\frac{567g^4S^5}{128m^8}+\frac{5103g^5S^6}{320m^{10}}+\cdots\right). \tag{4.11}$$

This result agrees with the one of [23], where the effective superpotentials were evaluated using both matrix model techinques and in terms of closed strings on Calabi-Yau geometry with fluxes.

4.2 Cubic superpotential

SO/Sp with traceful symmetric/antisymmetric matter. Now the superpotential under consideration takes the form:

$$W(\Phi) = \frac{m}{2}\,\mathrm{Tr}\,\Phi^2 + \frac{g}{3}\,\mathrm{Tr}\,\Phi^3\,. \tag{4.12}$$

The equation for $T(z)$ reads exactly as in (4.2) but now $c(z)$ and $f(z)$ are polynomials of degree 1. Again we factorize the denominator of both terms, now as follows:

$$W'(z)^2 + f(z) = g^2(z-k)^2(z+a+b)(z+a-b)\,. \tag{4.13}$$

In this case, the parameters a, b and k are complicated functions of m, g and f_1; because of this we will only write their expansion in powers of S:

$$\begin{aligned}
k &= -\frac{m}{g} + a \\
a &= \frac{g}{m^2}S + 3\frac{g^3}{m^5}S^2 + 16\frac{g^5}{m^8}S^3 + 105\frac{g^7}{m^{11}}S^4 + 768\frac{g^9}{m^{14}}S^5 + 6006\frac{g^{11}}{m^{17}}S^6 \\
&\quad +49152\frac{g^{13}}{m^{20}}S^7 + \cdots \\
b &= \sqrt{\frac{S}{2m}}\,\times \\
&\quad \times\left(2 + 2\frac{Sg^2}{m^3} + 9\frac{S^2g^4}{m^6} + 55\frac{S^3g^6}{m^9} + \frac{1547}{4}\frac{S^4g^8}{m^{12}} + \frac{11799}{4}\frac{S^5g^{10}}{m^{15}} + \frac{189805}{8}\frac{S^6g^{12}}{m^{18}} + \cdots\right).
\end{aligned} \tag{4.14}$$

Note that in the classical limit (that is $S \to 0$) the parameters a and b go to zero, while k tends to its classical value $-\frac{m}{g}$. Again the asymptotic behavior of $R(z)$ and $T(z)$ imposes $f_1 = -2gS$ and $c_1 = -4gN$, and, as before, c_0 is set by the condition that $T(z)$ does not have a pole at $z = k$:

$$\frac{1}{2\pi i}\oint_{C_k} dz\,T(z) = 0 \tag{4.15}$$

and from this

$$c_0 = 8\epsilon\left(2gk + g\sqrt{(a-b+k)\,(a+b+k)} + m\right) + 4gkN\,.$$

As before, $T(z)$ can be expanded in powers of $\frac{1}{z}$ and we can integrate the coefficient of $1/z^3$ with respect to m in order to obtain the effective superpotential.

We stress that without too much difficult one can obtain the result up to the desired order. For instance, up to seventh order:

$$\begin{aligned}
W_{\rm eff} &= \frac{1}{2}(-2\,\epsilon + N)\,S\log m - \frac{g^2}{2m^3}(-3\,\epsilon + N)\,S^2 - \frac{1}{12}\frac{g^4}{m^6}(-59\,\epsilon + 16N)\,S^3 - \\
&\quad -\frac{1}{24}\frac{g^6}{m^9}(-591\,\epsilon + 140N)\,S^4 - \frac{1}{16}\frac{g^8}{m^{12}}\left(-\frac{4775}{2}\,\epsilon + 512N\right)S^5 - \\
&\quad -\frac{1}{80}\frac{g^{10}}{m^{15}}(-80763\,\epsilon + 16016N)\,S^6 - \frac{1}{96}\frac{g^{12}}{m^{18}}(-704809\,\epsilon + 131072N)\,S^7 + \cdots\,.
\end{aligned} \tag{4.16}$$

Note that our results are in perfect agreement, up to S^5, with the ones of [21] found using the matrix model perturbative approach of [4] .

SO/Sp with traceless symmetric/antisymmetric matter. In the case of matter in the traceless representation, the equation of $T(z)$ reads:

$$
\begin{aligned}
T(z) = & -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}} + \\
& +2\,\epsilon\frac{\frac{d}{dz}\left(\left(W'(z)-\frac{1}{N}W'(\Phi)\right)-\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}\right)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}}\,. \qquad (4.17)
\end{aligned}
$$

Here, as we will see, the strategy we follow is different, due to the fact that we are considering the traceless representation and that $\mathrm{Tr}(\Phi^2)$ appears explicitly in the denominator of $T(z)$. First we factorize the denominator in the usual way:

$$
\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)=g^2(z-k)^2\left(z^2+az+b\right). \qquad (4.18)
$$

As we are in the traceless representation we have:

$$
\begin{aligned}
\mathrm{Tr}\,\Phi &= 0 \\
W'(\Phi) &= g\,\mathrm{Tr}\,\Phi^2\,. \qquad (4.19)
\end{aligned}
$$

The polynomial $c(z)$ can be fixed as before, and again $f_1=-2gS$. The condition of $\mathrm{Tr}\,\Phi=0$ implies that the coefficient of $\frac{1}{z^2}$ in the expansion of $T(z)$ must be zero. We can use this condition together with the conditions of factorization in order to obtain a system of equations from where $\mathrm{Tr}\,\Phi^2$ can be evaluated. Equivalently the traceless condition can be used to determine c_0 and equation (4.15) together with the conditions from factorization can be used to determine $\mathrm{Tr}\,\Phi^2$. As before we obtain from this the effective superpotential. It should be stressed that such evaluation can be done at any desired number of loops, without many technical complications. For the effective superpotential one finds:

$$
\begin{aligned}
W_{\mathrm{eff}} = & (N-2\epsilon)\frac{S}{2}\log m+\frac{g^2(-\epsilon\,N+4)S^2}{2Nm^3}+ \\
& +\frac{g^4(160\,\epsilon-24N-N^2\,\epsilon)S^3}{12m^6N^2}+\frac{g^6(3584-256\,\epsilon N-36N^2-\epsilon\,N^3)S^4}{24m^9N^3}+ \\
& +\frac{g^8\left(67584\,\epsilon-704N^2\,\epsilon-48N^3-N^4\,\epsilon\right)S^5}{32m^{12}N^4}+ \qquad (4.20) \\
& +\frac{7g^{10}\left(1171456+79872\,\epsilon N-8320N^2-1280\epsilon N^3-60N^4-\epsilon N^5\right)S^6}{240m^{15}N^5}+\cdots\,.
\end{aligned}
$$

Note that these results agree with the ones of [21]; however using this method is easier to compute higher loop corrections.

$\mathrm{SU}(N)$ **with adjoint matter.** Recently in [21] it was found that for a cubic potential, like (4.12), the perturbative part of the effective superpotential is zero up to terms of order S^4, due to cancellations in the diagrammatic evaluation. In this paragraph we will show

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that the generalized Konishi anomaly implies that the perturbative part of W_{eff} is exactly vanishing to all orders. Let us consider equation (3.16) and expand it in powers of $1/z$

$$\begin{aligned} T(z) &= -\frac{1}{4}\frac{c_0+c_1 z}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}} \\ &= -\frac{c_1}{4g}\frac{1}{z}+\frac{1}{4g}\left(-c_0+\frac{c_1 m}{g}\right)\frac{1}{z^2}+\mathcal{O}\left(\frac{1}{z^3}\right) \\ &= \frac{N}{z}+\frac{\langle \mathrm{Tr}\,\Phi\rangle}{z^2}+\mathcal{O}\left(\frac{1}{z^3}\right). \end{aligned} \tag{4.21}$$

From the terms of order $1/z$ we find the familiar condition $c_1=-4gN$. Considering the term $1/z^2$ and imposing the tracelessness of Φ in $\mathrm{SU}(N)$ we obtain the relation $c_0=\frac{c_1 m}{g}$. Again the denominator of (4.21) can be factorized as in (4.18)

$$\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)=g^2(z-k)^2\left(z^2+az+b\right)$$

then the condition (4.15) gives the following relation

$$a=0\,.$$

With this condition only odd powers of $\frac{1}{z}$ will be present in the expansion of $T(z)$; in particular

$$\frac{\partial W_{\text{eff}}}{\partial g}=0$$

from which we see that the perturbative part of the effective superpotential is identically zero (remember that the perturbative part depends only on a specific ratio of m and g, in this case g^2/m^3).

We stress that the vanishing of the superpotential is a particular characteristic of the cubic superpotential. One can easily check that for a quartic tree level superpotential, a non zero result is obtained.

5. Conclusions

In this paper we used the generalized Konishi anomaly approach of [5] in order to compute some effective superpotentials for the gauge groups $\mathrm{SO}(N)$, $\mathrm{Sp}(N)$ and $\mathrm{SU}(N)$ with quartic and cubic tree level superpotential and matter in various representations. Our results are summarized in the appendix.

Our results agree with previous literature, where superpotentials were computed summing planar diagrams in the matrix model context. The method used in this paper does not rely on any diagrammatic expansion, but simply on Ward identities that allow us to write closed expressions for the generating functions of correlators; because of this higher corrections are easily evaluated by power expansions without having to draw any diagram.

In particular, the cancellation of the perturbative part of W_{eff} for $\mathrm{SU}(N)$ gauge group and cubic superpotential, that in [21] was checked diagrammatically up to fourth order in S, can be easily shown to hold at any order.

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In [21], it was found within the matrix model approach, for the $\mathrm{Sp}(N)$ gauge theory, a result disagreeing with previous gauge theory computations [24] at $h = \frac{N}{2} + 1$ loops, h being the dual Coxeter number. This led the authors of [21] to suggest that the general Dijkgraaf-Vafa conjecture needs to be modified, at least in this particular case. In this paper we reproduced their matrix model prediction from the gauge theory side, within the formalism of [5]. Then, unfortunatly, we are not able to give any hint on this intriguing problem.

Possible further developments are to apply these techniques to more general superpotentials or other gauge groups (maybe exceptional); in some cases it may be even possible to find the *exact* effective superpotential.

Acknowledgments

We are particularly grateful to E. Gava and K. S. Narain for useful discussions and comments. We also thank J. David, R. Dijkgraaf, C. Maccaferri, D. Mamone, and L. Mazzucato for discussions.

A. Summary of results

In this appendix we collect our result, stating first the equation for $T(z)$ and then the corresponding superpotential for each case. Remember that $\epsilon = \pm 1$ for $\mathrm{Sp}(N)/\mathrm{SO}(N)$.

A.1 $\mathrm{SU}(N)$ with adjoint matter and cubic interactions

$$\begin{aligned} T(z) &= -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}} \\ \frac{\partial W_{\mathrm{eff}}}{\partial g} &= 0\,. \end{aligned} \tag{A.1}$$

A.2 $\mathrm{Sp}(N)/\mathrm{SO}(N)$ with matter in the antisymmetric/symmetric traceful representation

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} + 2\epsilon\frac{W''(z) - \left(\sqrt{W'(z)^2 + f(z)}\right)'}{\sqrt{W'(z)^2 + f(z)}}\,.$$

For a cubic superpotential

$$\begin{aligned} W_{\mathrm{eff}} &= \frac{1}{2}\left(-2\,\epsilon + N\right) S \log m - \frac{g^2}{2m^3}\left(-3\,\epsilon + N\right) S^2 - \frac{1}{12}\frac{g^4}{m^6}\left(-59\,\epsilon + 16N\right) S^3 - \\ &\quad - \frac{1}{24}\frac{g^6}{m^9}\left(-591\,\epsilon + 140N\right) S^4 - \frac{1}{16}\frac{g^8}{m^{12}}\left(-\frac{4775}{2}\,\epsilon + 512N\right) S^5 - \\ &\quad - \frac{1}{80}\frac{g^{10}}{m^{15}}\left(-80763\,\epsilon + 16016N\right) S^6 - \frac{1}{96}\frac{g^{12}}{m^{18}}\left(-704809\,\epsilon + 131072N\right) S^7 + \cdots \end{aligned} \tag{A.2}$$

For a quartic superpotential

$$\begin{aligned} W_{\mathrm{eff}} &= \frac{1}{2}\left(-2\,\epsilon + N\right) S \log m + \frac{g}{8m^2}\left(-10\,\epsilon + 3N\right) S^2 - \frac{g^2}{16m^4}\left(-38\,\epsilon + 9N\right) S^3 \\ &\quad + \frac{g^3}{96m^6}\left(-662\,\epsilon + 135N\right) S^4 + \cdots\,. \end{aligned} \tag{A.3}$$

A.3 Sp(N)/SO(N) with matter in the antisymmetric/symmetric traceless representation and cubic interactions

$$
\begin{aligned}
T(z) &= -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}} + \\
&\quad +2\,\epsilon\frac{\frac{d}{dz}\left(\left(W'(z)-\frac{1}{N}W'(\Phi)\right)-\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}\right)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}}
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
W_{\mathrm{eff}} &= (N-2\epsilon)\frac{S}{2}\log m + \frac{g^2(-\epsilon\,N+4)S^2}{2Nm^3} \\
&\quad + \frac{g^4(160\,\epsilon-24N-N^2\,\epsilon)S^3}{12m^6N^2} + \frac{g^6(3584-256\,\epsilon N-36N^2-\epsilon\,N^3)S^4}{24m^9N^3} \\
&\quad + \frac{g^8\left(67584\,\epsilon-704N^2\,\epsilon-48N^3-N^4\,\epsilon\right)S^5}{32m^{12}N^4} \\
&\quad + \frac{7g^{10}\left(1171456+79872\,\epsilon N-8320N^2-1280\epsilon N^3-60N^4-\epsilon N^5\right)S^6}{240m^{15}N^5} + \cdots\,.
\end{aligned}
\tag{A.5}
$$

A.4 Sp(N)/SO(N) with matter in the adjoint representation and quartic interactions

$$
T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}} + \epsilon\frac{2}{z}\frac{W'(z)-\sqrt{W'(z)^2+f(z)}}{\sqrt{W'(z)^2+f(z)}}
\tag{A.6}
$$

$$
W_{\mathrm{eff}} = (N+2\epsilon)\left(\frac{S}{2}\log m + \frac{3gS^2}{8m^2} - \frac{9g^2S^3}{16m^4} + \frac{45g^3S^4}{32m^6} - \frac{567g^4S^5}{128m^8} + \frac{5103g^5S^6}{320m^{10}} + \cdots\right).
$$

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